HANOI MATHEMATICAL SOCIETY

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HANOI OPEN MATHEMATICS COMPETITON PROBLEMS 2006 - 2013

HANOI - 2013

Contents

1 Hanoi Open Mathematics Competition **3**

1.1 Hanoi Open Mathematics Competition 2006 . . . 3

1.1.1 Junior Section 3

1.1.2 Senior Section 4

1.2 Hanoi Open Mathematics Competition 2007 . . . 6

1.2.1 Junior Section 6

1.2.2 Senior Section 8

1.3 Hanoi Open Mathematics Competition 2008 . . . 10

1.3.1 Junior Section 10

1.3.2 Senior Section 11

1.4 Hanoi Open Mathematics Competition 2009 . . . 13

1.4.1 Junior Section 13

1.4.2 Senior Section 15

1.5 Hanoi Open Mathematics Competition 2010 . . . 16

1.5.1 Junior Section 16

1.5.2 Senior Section 18

1.6 Hanoi Open Mathematics Competition 2011 . . . 19

1.6.1 Junior Section 19

1.6.2 Senior Section 21

1.7 Hanoi Open Mathematics Competition 2012 . . . 23

1.7.1 Junior Section 23
1.7.2 Senior Section 26
1.8 Hanoi Open Mathematics Competition 2013 . . . 28
1.8.1 Junior Section 28
1.8.2 Senior Section 31

Chapter 1

Hanoi Open Mathematics Competition

1.1 Hanoi Open Mathematics Competition 2006

1.1.1 Junior Section

Question 1. What is the last two digits of the number

$$(11 + 12 + 13 + \cdots + 2006)^2?$$

Question 2. Find the last two digits of the sum

$$2005^{11} + 2005^{12} + \cdots + 2005^{2006}.$$

Question 3. Find the number of different positive integer triples (x, y, z) satisfying the equations

$$x^2 + y - z = 100 \quad \text{and} \quad x + y^2 - z = 124.$$

Question 4. Suppose x and y are two real numbers such that

$$x + y - xy = 155 \quad \text{and} \quad x^2 + y^2 = 325.$$

Find the value of $|x^3 - y^3|$.

Question 5. Suppose n is a positive integer and 3 arbitrary numbers are choosen from the set $\{1, 2, 3, \ldots, 3n+1\}$ with their sum equal to $3n + 1$.

What is the largest possible product of those 3 numbers?

Question 6. The figure $ABCDEF$ is a regular hexagon. Find all points M belonging to the hexagon such that

$$\text{Area of triangle } MAC = \text{Area of triangle } MCD.$$

Question 7. On the circle (O) of radius 15cm are given 2 points A, B. The altitude OH of the triangle OAB intersect (O) at C. What is AC if $AB = 16cm$?

Question 8. In ΔABC, $PQ \parallel BC$ where P and Q are points on AB and AC respectively. The lines PC and QB intersect at G. It is also given $EF//BC$, where $G \in EF$, $E \in AB$ and $F \in AC$ with $PQ = a$ and $EF = b$. Find value of BC.

Question 9. What is the smallest possible value of

$$x^2 + y^2 - x - y - xy?$$

1.1.2 Senior Section

Question 1. What is the last three digits of the sum

$$11! + 12! + 13! + \cdots + 2006!$$

Question 2. Find the last three digits of the sum

$$2005^{11} + 2005^{12} + \cdots + 2005^{2006}.$$

Question 3. Suppose that

$$a^{\log_b c} + b^{\log_c a} = m.$$

Find the value of

$$c^{\log_b a} + a^{\log_c b}?$$

Question 4. Which is larger

$$2^{\sqrt{2}}, \quad 2^{1+\frac{1}{\sqrt{2}}} \quad \text{and} \quad 3.$$

Question 5. The figure $ABCDEF$ is a regular hexagon. Find all points M belonging to the hexagon such that

$$\text{Area of triangle } MAC = \text{Area of triangle } MCD.$$

Question 6. On the circle of radius 30cm are given 2 points A, B with $AB = 16$cm and C is a midpoint of AB. What is the perpendicular distance from C to the circle?

Question 7. In ΔABC, $PQ \parallel BC$ where P and Q are points on AB and AC respectively. The lines PC and QB intersect at G. It is also given $EF//BC$, where $G \in EF$, $E \in AB$ and $F \in AC$ with $PQ = a$ and $EF = b$. Find value of BC.

Question 8. Find all polynomials $P(x)$ such that

$$P(x) + \left(\frac{1}{x}\right) = x + \frac{1}{x}, \quad \forall x \neq 0.$$

Question 9. Let x, y, z be real numbers such that $x^2+y^2+z^2 = 1$. Find the largest possible value of

$$|x^3 + y^3 + z^3 - xyz|?$$

1.2 Hanoi Open Mathematics Competition 2007

1.2.1 Junior Section

Question 1. What is the last two digits of the number

$$(3 + 7 + 11 + \cdots + 2007)^2?$$

(A) 01; (B) 11; (C) 23; (D) 37; (E) None of the above.

Question 2. What is largest positive integer n satisfying the following inequality:

$n^{2006} < 7^{2007}$?

(A) 7; (B) 8; (C) 9; (D) 10; (E) 11.

Question 3. Which of the following is a possible number of diagonals of a convex polygon?

(A) 02; (B) 21; (C) 32; (D) 54; (E) 63.

Question 4. Let m and n denote the number of digits in 2^{2007} and 5^{2007} when expressed in base 10. What is the sum $m + n$?

(A) 2004; (B) 2005; (C) 2006; (D) 2007; (E) 2008.

Question 5. Let be given an open interval $(\alpha; eta)$ with $eta - \alpha = \frac{1}{2007}$. Determine the maximum number of irreducible fractions $\frac{a}{b}$ in $(\alpha; eta)$ with $1 \leq b \leq 2007$?

(A) 1002; (B) 1003; (C) 1004; (D) 1005; (E) 1006.

Question 6. In triangle ABC, $\angle BAC = 60^0$, $\angle ACB = 90^0$ and D is on BC. If AD bisects $\angle BAC$ and $CD = 3cm$. Then DB is

(A) 3; (B) 4; (C) 5; (D) 6; (E) 7.

Question 7. Nine points, no three of which lie on the same straight line, are located inside an equilateral triangle of side 4. Prove that some three of these points are vertices of a triangle whose area is not greater than $\sqrt{3}$.

Question 8. Let a, b, c be positive integers. Prove that

$$\frac{(b+c-a)^2}{(b+c)^2+a^2}+\frac{(c+a-b)^2}{(c+a)^2+b^2}+\frac{(a+b-c)^2}{(a+b)^2+c^2}\geq\frac{3}{5}.$$

Question 9. A triangle is said to be the Heron triangle if it has integer sides and integer area. In a Heron triangle, the sides a, b, c satisfy the equation $b = a(a-c)$.

Prove that the triangle is isosceles.

Question 10. Let a, b, c be positive real numbers such that $\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}\geq 1$. Prove that $\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}\geq 1$.

Question 11. How many possible values are there for the sum $a+b+c+d$ if a, b, c, d are positive integers and $abcd = 2007$.

Question 12. Calculate the sum

$$\frac{5}{2.7}+\frac{5}{7.12}+\cdots+\frac{5}{2002.2007}.$$

Question 13. Let be given triangle ABC. Find all points M such that area of $\Delta MAB=$ area of ΔMAC.

Question 14. How many ordered pairs of integers (x, y) satisfy the equation

$$2x^2+y^2+xy=2(x+y)?$$

Question 15. Let $p=\overline{abc}$ be the 3-digit prime number. Prove that the equation

$$ax^2+bx+c=0$$

has no rational roots.

1.2.2 Senior Section

Question 1. What is the last two digits of the number

$$\left(11^2 + 15^2 + 19^2 + \cdots + 2007^2\right)^2?$$

(A) 01; (B) 21; (C) 31; (D) 41; (E) None of the above.

Question 2. Which is largest positive integer n satisfying the following inequality:

$$n^{2007} > (2007)^n.$$

(A) 1; (B) 2; (C) 3; (D) 4; (E) None of the above.

Question 3. Find the number of different positive integer triples (x, y, z) satsfying the equations

$$x + y - z = 1 \quad \text{and} \quad x^2 + y^2 - z^2 = 1.$$

(A) 1; (B) 2; (C) 3; (D) 4; (E) None of the above.

Question 4. List the numbers $\sqrt{2}$, $\sqrt[3]{3}$, , $\sqrt[4]{4}$, $\sqrt[5]{5}$ and $\sqrt[6]{6}$ in order from greatest to least.

Question 5. Suppose that A, B, C, D are points on a circle, AB is the diameter, CD is perpendicular to AB and meets AB at E, AB and CD are integers and $AE - EB = \sqrt{3}$.

Find AE?

Question 6. Let $P(x) = x^3 + ax^2 + bx + 1$ and $|P(x)| \leq 1$ for all x such that $|x| \leq 1$. Prove that $|a| + |b| \leq 5$.

Question 7. Find all sequences of integers $x_1, x_2, \ldots, x_n, \ldots$ such that ij divides $x_i + x_j$ for any two distinct positive integers i and j.

Question 8.

Let ABC be an equilateral triangle. For a point M inside ΔABC, let D, E, F be the feet of the perpendiculars from M onto BC, CA, AB, respectively. Find the locus of all such points M for which $\angle FDE$ is a right angle.

Question 9. Let $a_1, a_2, \ldots, a_{2007}$ be real numbers such that

$$a_1+a_2+\cdots+a_{2007} \geq (2007)^2 \text{ and } a_1^2+a_2^2+\cdots+a_{2007}^2 \leq (2007)^3-1.$$

Prove that $a_k \in [2006; 2008]$ for all $k \in \{1, 2, \ldots, 2007\}$.

Question 10. What is the smallest possible value of

$$x^2 + 2y^2 - x - 2y - xy?$$

Question 11. Find all polynomials $P(x)$ satisfying the equation

$$(2x-1)P(x) = (x-1)P(2x), \ \forall x.$$

Question 12. Calculate the sum

$$\frac{1}{2.7.12} + \frac{1}{7.12.17} + \cdots + \frac{1}{1997.2002.2007}.$$

Question 13. Let ABC be an acute-angle triangle with $BC > CA$. Let O, H and F be the circumcenter, orthocentre and the foot of its altitude CH, respectively.

Suppose that the perpendicular to OF at F meet the side CA at P. Prove $\angle FHP = \angle BAC$.

Question 14. How many ordered pairs of integers (x, y) satisfy the equation

$$x^2 + y^2 + xy = 4(x + y)?$$

Question 15. Let $p = \overline{abcd}$ be the 4-digit prime number. Prove that the equation

$$ax^3 + bx^2 + cx + d = 0$$

has no rational roots.

1.3 Hanoi Open Mathematics Competition 2008

1.3.1 Junior Section

Question 1. How many integers from 1 to 2008 have the sum of their digits divisible by 5 ?

Question 2. How many integers belong to $(a, 2008a)$, where a $(a > 0)$ is given.

Question 3. Find the coefficient of x in the expansion of

$$(1 + x)(1 - 2x)(1 + 3x)(1 - 4x) \cdots (1 - 2008x).$$

Question 4. Find all pairs (m, n) of positive integers such that

$$m^2 + n^2 = 3(m + n).$$

Question 5. Suppose x, y, z, t are real numbers such that

$$\begin{cases} |x + y + z - t| \leqslant 1 \\ |y + z + t - x| \leqslant 1 \\ |z + t + x - y| \leqslant 1 \\ |t + x + y - z| \leqslant 1 \end{cases}$$

Prove that $x^2 + y^2 + z^2 + t^2 \leqslant 1$.

Question 6. Let $P(x)$ be a polynomial such that

$$P(x^2 - 1) = x^4 - 3x^2 + 3.$$

Find $P(x^2+1)$?

Question 7. The figure $ABCDE$ is a convex pentagon. Find the sum

$$\angle DAC + \angle EBD + \angle ACE + \angle BDA + \angle CEB?$$

Question 8. The sides of a rhombus have length a and the area is S. What is the length of the shorter diagonal?

Question 9. Let be given a right-angled triangle ABC with $\angle A = 90^0$, $AB = c$, $AC = b$. Let $E \in AC$ and $F \in AB$ such that $\angle AEF = \angle ABC$ and $\angle AFE = \angle ACB$. Denote by $P \in BC$ and $Q \in BC$ such that $EP \perp BC$ and $FQ \perp BC$. Determine $EP + EF + PQ$?

Question 10. Let $a, b, c \in [1, 3]$ and satisfy the following conditions

$$\max\{a, b, c\} \geqslant 2,\ a + b + c = 5.$$

What is the smallest possible value of

$$a^2 + b^2 + c^2?$$

1.3.2 Senior Section

Question 1. How many integers are there in $(b, 2008b]$, where b $(b > 0)$ is given.

Question 2. Find all pairs (m, n) of positive integers such that

$$m^2 + 2n^2 = 3(m + 2n).$$

Question 3. Show that the equation

$$x^2 + 8z = 3 + 2y^2$$

has no solutions of positive integers x, y and z.

Question 4. Prove that there exists an infinite number of relatively prime pairs (m, n) of positive integers such that the equation

$$x^3 - nx + mn = 0$$

has three distint integer roots.

Question 5. Find all polynomials $P(x)$ of degree 1 such that

$$\max_{a \leq x \leq b} P(x) - \min_{a \leq x \leq b} P(x) = b - a, \ \forall a, b \in \mathbb{R} \ \text{ where } \ a < b.$$

Question 6. Let $a, b, c \in [1, 3]$ and satisfy the following conditions

$$\max\{a, b, c\} \geqslant 2, \ \ a + b + c = 5.$$

What is the smallest possible value of

$$a^2 + b^2 + c^2?$$

Question 7. Find all triples (a, b, c) of consecutive odd positive integers such that $a < b < c$ and $a^2 + b^2 + c^2$ is a four digit number with all digits equal.

Question 8. Consider a convex quadrilateral $ABCD$. Let O be the intersection of AC and BD; M, N be the centroid of $\triangle AOB$ and $\triangle COD$ and P, Q be orthocenter of $\triangle BOC$ and $\triangle DOA$, respectively. Prove that $MN \perp PQ$.

Question 9. Consider a triangle ABC. For every point $M \in BC$ we difine $N \in CA$ and $P \in AB$ such that $APMN$ is a parallelogram. Let O be the intersection of BN and CP. Find $M \in BC$ such that $\angle PMO = \angle OMN$.

Question 10. Let be given a right-angled triangle ABC with $\angle A = 90^0$, $AB = c$, $AC = b$. Let $E \in AC$ and $F \in AB$ such that $\angle AEF = \angle ABC$ and $\angle AFE = \angle ACB$. Denote by $P \in BC$ and $Q \in BC$ such that $EP \perp BC$ and $FQ \perp BC$. Determine $EP + EF + FQ$?

1.4 Hanoi Open Mathematics Competition 2009

1.4.1 Junior Section

Question 1. Let a, b, c be 3 distinct numbers from $\{1, 2, 3, 4, 5, 6\}$. Show that 7 divides $abc + (7 - a)(7 - b)(7 - c)$.

Question 2. Show that there is a natural number n such that the number $a = n!$ ends exacly in 2009 zeros.

Question 3. Let a, b, c be positive integers with no common factor and satisfy the conditions

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c}.$$

Prove that $a + b$ is a square.

Question 4. Suppose that $a = 2^b$, where $b = 2^{10n+1}$. Prove that a is divisible by 23 for any positive integer n.

Question 5. Prove that $m^7 - m$ is divisible by 42 for any positive integer m.

Question 6. Suppose that 4 real numbers a, b, c, d satisfy the conditions

$$\begin{cases} a^2 + b^2 = 4 \\ c^2 + d^2 = 4 \\ ac + bd = 2 \end{cases}$$

Find the set of all possible values the number $M = ab + cd$ can take.

Question 7. Let a, b, c, d be positive integers such that $a + b + c + d = 99$. Find the smallest and the greatest values of the following product $P = abcd$.

Question 8. Find all the pairs of the positive integers such that the product of the numbers of any pair plus the half of one of the numbers plus one third of the other number is three times less than 1004.

Question 9. Let be given ΔABC with area $(\Delta ABC) = 60\text{cm}^2$. Let R, S lie in BC such that $BR = RS = SC$ and P, Q be midpoints of AB and AC, respectively. Suppose that PS intersects QR at T. Evaluate area (ΔPQT).

Question 10. Let ABC be an acute-angled triangle with $AB = 4$ and CD be the altitude through C with $CD = 3$. Find the distance between the midpoints of AD and BC.

Question 11. Let $A = \{1, 2, \ldots, 100\}$ and B is a subset of A having 48 elements. Show that B has two distint elements x and y whose sum is divisible by 11.

1.4.2 Senior Section

Question 1. Let a, b, c be 3 distinct numbers from $\{1, 2, 3, 4, 5, 6\}$. Show that 7 divides $abc + (7 - a)(7 - b)(7 - c)$.

Question 2. Show that there is a natural number n such that the number $a = n!$ ends exacly in 2009 zeros.

Question 3. Let a, b, c be positive integers with no common factor and satisfy the conditions

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c}.$$

Prove that $a + b$ is a square.

Question 4. Suppose that $a = 2^b$, where $b = 2^{10n+1}$. Prove that a is divisible by 23 for any positive integer n.

Question 5. Prove that $m^7 - m$ is divisible by 42 for any positive integer m.

Question 6. Suppose that 4 real numbers a, b, c, d satisfy the conditions

$$\begin{cases} a^2 + b^2 = 4 \\ c^2 + d^2 = 4 \\ ac + bd = 2 \end{cases}$$

Find the set of all possible values the number $M = ab + cd$ can take.

Question 7. Let a, b, c, d be positive integers such that $a + b + c + d = 99$. Find the smallest and the greatest values of the following product $P = abcd$.

Question 8. Find all the pairs of the positive integers such that the product of the numbers of any pair plus the half of one of the numbers plus one third of the other number is three times less than 1004.

Question 9.Given an acute-angled triangle ABC with area S, let points A', B', C' be located as follows: A' is the point where altitude from A on BC meets the outwards facing semicirle drawn on BC as diameter. Points B', C' are located similarly. Evaluate the sum

$$T = (\text{area } \Delta BCA')^2 + (\text{area } \Delta CAB')^2 + (\text{area } \Delta ABC')^2.$$

Question 10. Prove that $d^2+(a-b)^2 < c^2$, where d is diameter of the inscribed circle of ΔABC.

Question 11. Let $A = \{1, 2, \ldots, 100\}$ and B is a subset of A having 48 elements. Show that B has two distint elements x and y whose sum is divisible by 11.

1.5 Hanoi Open Mathematics Competition 2010

1.5.1 Junior Section

Question 1. Compare the numbers:

$$P = \underbrace{888\ldots888}_{2010\ digits} \times \underbrace{333\ldots333}_{2010\ digits} \text{ and } Q = \underbrace{444\ldots444}_{2010\ digits} \times \underbrace{666\ldots667}_{2010\ digits}$$

(A): $P = Q$; (B): $P > Q$; (C): $P < Q$.

Question 2. The number of integer n from the set $\{2000, 2001, \ldots, 2010\}$ such that $2^{2n} + 2^n + 5$ is divisible by 7:

(A): 0; (B): 1; (C): 2; (D): 3; (E) None of the above.

Question 3. 5 last digits of the number $M = 5^{2010}$ are

(A): 65625; (B): 45625; (C): 25625; (D): 15625; (E) None of the above.

Question 4. How many real numbers $a \in (1, 9)$ such that the corresponding number $a - \dfrac{1}{a}$ is an integer.

(A): 0; (B): 1; (C): 8; (D): 9; (E) None of the above.

Question 5. Each box in a 2×2 table can be colored black or white. How many different colorings of the table are there?

(A): 4; (B): 8; (C): 16; (D): 32; (E) None of the above.

Question 6. The greatest integer less than $(2 + \sqrt{3})^5$ are

(A): 721; (B): 722; (C): 723; (D): 724; (E) None of the above.

Question 7. Determine all positive integer a such that the equation

$$2x^2 - 30x + a = 0$$

has two prime roots, i.e. both roots are prime numbers.

Question 8. If n and $n^3 + 2n^2 + 2n + 4$ are both perfect squares, find n.

Question 9. Let be given a triangle ABC and points D, M, N belong to BC, AB, AC, respectively. Suppose that MD is parallel to AC and ND is parallel to AB. If $S_{\Delta BMD} = 9\text{cm}^2$, $S_{\Delta DNC} = 25\text{cm}^2$, compute $S_{\Delta AMN}$?

Question 10. Find the maximum value of

$$M = \frac{x}{2x+y} + \frac{y}{2y+z} + \frac{z}{2z+x}, \quad x, y, z > 0.$$

1.5.2 Senior Section

Question 1. The number of integers $n \in [2000, 2010]$ such that $2^{2n} + 2^n + 5$ is divisible by 7 is

(A): 0; (B): 1; (C): 2; (D): 3; (E) None of the above.

Question 2. 5 last digits of the number 5^{2010} are

(A): 65625; (B): 45625; (C): 25625; (D): 15625; (E) None of the above.

Question 3. How many real numbers $a \in (1, 9)$ such that the corresponding number $a - \dfrac{1}{a}$ is an integer.

(A): 0; (B): 1; (C): 8; (D): 9; (E) None of the above.

Question 4. Each box in a 2×2 table can be colored black or white. How many different colorings of the table are there?

Question 5. Determine all positive integer a such that the equation

$$2x^2 - 30x + a = 0$$

has two prime roots, i.e. both roots are prime numbers.

Question 6. Let a, b be the roots of the equation $x^2 - px + q = 0$ and let c, d be the roots of the equation $x^2 - rx + s = 0$, where p, q, r, s are some positive real numbers. Suppose that

$$M = \frac{2(abc + bcd + cda + dab)}{p^2 + q^2 + r^2 + s^2}$$

is an integer. Determine a, b, c, d.

Question 7. Let P be the common point of 3 internal bisectors of a given ABC. The line passing through P and perpendicular to CP intersects AC and BC at M and N, respectively. If $AP = 3$cm, $BP = 4$cm, compute the value of $\dfrac{AM}{BN}$?

Question 8. If n and n^3+2n^2+2n+4 are both perfect squares, find n.

Question 9. Let x, y be the positive integers such that $3x^2 + x = 4y^2 + y$. Prove that $x - y$ is a perfect integer.

Question 10. Find the maximum value of

$$M = \frac{x}{2x+y} + \frac{y}{2y+z} + \frac{z}{2z+x}, \quad x, y, z > 0.$$

1.6 Hanoi Open Mathematics Competition 2011

1.6.1 Junior Section

Question 1. Three lines are drawn in a plane. Which of the following could NOT be the total number of points of intersections?

(A): 0; (B): 1; (C): 2; (D): 3; (E): They all could.

Question 2. The last digit of the number $A = 7^{2011}$ is

(A) 1; (B) 3; (C) 7; (D) 9; (E) None of the above.

Question 3. What is the largest integer less than or equal to

$$\sqrt[3]{(2011)^3 + 3\times(2011)^2 + 4\times 2011 + 5}?$$

(A) 2010; (B) 2011; (C) 2012; (D) 2013; (E) None of the above.

Question 4. Among the four statements on real numbers below, how many of them are correct?

"If $a < b < 0$ then $a < b^2$";
"If $0 < a < b$ then $a < b^2$";
"If $a^3 < b^3$ then $a < b$";
"If $a^2 < b^2$ then $a < b$";
"If $|a| < |b|$ then $a < b$".

(A) 0; (B) 1; (C) 2; (D) 3; (E) 4

Question 5. Let $M = 7! \times 8! \times 9! \times 10! \times 11! \times 12!$. How many factors of M are perfect squares?

Question 6. Find all positive integers (m, n) such that

$$m^2 + n^2 + 3 = 4(m + n).$$

Question 7. Find all pairs (x, y) of real numbers satisfying the system

$$\begin{cases} x + y = 3 \\ x^4 - y^4 = 8x - y \end{cases}$$

Question 8. Find the minimum value of

$$S = |x + 1| + |x + 5| + |x + 14| + |x + 97| + |x + 1920|.$$

Question 9. Solve the equation

$$1 + x + x^2 + x^3 + \cdots + x^{2011} = 0.$$

Question 10. Consider a right-angle triangle ABC with $A = 90^o$, $AB = c$ and $AC = b$. Let $P \in AC$ and $Q \in AB$ such that $\angle APQ = \angle ABC$ and $\angle AQP = \angle ACB$. Calculate $PQ + PE + QF$, where E and F are the projections of P and Q onto BC, respectively.

Question 11. Given a quadrilateral $ABCD$ with $AB = BC = 3$cm, $CD = 4$cm, $DA = 8$cm and $\angle DAB + \angle ABC = 180^o$. Calculate the area of the quadrilateral.

Question 12. Suppose that $a > 0, b > 0$ and $a + b \leqslant 1$. Determine the minimum value of

$$M = \frac{1}{ab} + \frac{1}{a^2 + ab} + \frac{1}{ab + b^2} + \frac{1}{a^2 + b^2}.$$

1.6.2 Senior Section

Question 1. An integer is called "octal" if it is divisible by 8 or if at least one of its digits is 8. How many integers between 1 and 100 are octal?

(A): 22; (B): 24; (C): 27; (D): 30; (E): 33.

Question 2. What is the smallest number

(A) 3; (B) $2^{\sqrt{2}}$; (C) $2^{1+\frac{1}{\sqrt{2}}}$; (D) $2^{\frac{1}{2}} + 2^{\frac{2}{3}}$; (E) $2^{\frac{5}{3}}$.

Question 3. What is the largest integer less than to

$$\sqrt[3]{(2011)^3 + 3 \times (2011)^2 + 4 \times 2011 + 5}?$$

(A) 2010; (B) 2011; (C) 2012; (D) 2013; (E) None of the above.

Question 4. Prove that

$$1 + x + x^2 + x^3 + \cdots + x^{2011} \geqslant 0$$

for every $x \geqslant -1$.

Question 5. Let a, b, c be positive integers such that $a + 2b + 3c = 100$. Find the greatest value of $M = abc$.

Question 6. Find all pairs (x, y) of real numbers satisfying the system

$$\begin{cases} x + y = 2 \\ x^4 - y^4 = 5x - 3y \end{cases}$$

Question 7. How many positive integers a less than 100 such that $4a^2 + 3a + 5$ is divisible by 6.

Question 8. Find the minimum value of

$$S = |x + 1| + |x + 5| + |x + 14| + |x + 97| + |x + 1920|.$$

Question 9. For every pair of positive integers $(x; y)$ we define $f(x; y)$ as follows:

$$f(x, 1) = x$$
$$f(x, y) = 0 \quad \text{if} \quad y > x$$
$$f(x + 1, y) = y[f(x, y) + f(x, y - 1)]$$

Evaluate $f(5; 5)$.

Question 10. Two bisectors BD and CE of the triangle ABC intersect at O. Suppose that $BD.CE = 2BO.OC$. Denote by H the point in BC such that $OH \perp BC$. Prove that $AB.AC = 2HB.HC$.

Question 11. Consider a right-angle triangle ABC with $A = 90^o$, $AB = c$ and $AC = b$. Let $P \in AC$ and $Q \in AB$ such that $\angle APQ = \angle ABC$ and $\angle AQP = \angle ACB$. Calculate $PQ + PE + QF$, where E and F are the projections of P and Q onto BC, respectively.

Question 12. Suppose that $|ax^2 + bx + c| \geqslant |x^2 - 1|$ for all real numbers x. Prove that $|b^2 - 4ac| \geqslant 4$.

1.7 Hanoi Open Mathematics Competition 2012

1.7.1 Junior Section

Question 1. Assume that $a - b = -(a - b)$. Then:

(A) $a = b$; (B) $a < b$; (C) $a > b$; (D); It is impossible to compare those of a and b.

Question 2. Let be given a parallelogram $ABCD$ with the area of 12cm^2. The line through A and the midpoint M of BC meets BD at N. Compute the area of the quadrilateral $MNDC$.
(A): 4cm^2; (B): 5cm^2; (C): 6cm^2; (D): 7cm^2; (E) None of the above.

Question 3. For any possitive integer a, let $[a]$ denote the smallest prime factor of a. Which of the following numbers is equal to $[35]$?
(A) $[10]$; (B) $[15]$; (C) $[45]$; (D) $[55]$; (E) $[75]$;

Question 4. A man travels from town A to town E through towns B, C and D with uniform speeds 3km/h, 2km/h, 6km/h and 3km/h on the horizontal, up slope, down slope and horizontal road, respectively. If the road between town A and town E can be classified as horizontal, up slope, down slope and horizontal and total length of each type of road is the same, what is the average speed of his journey?

(A) 2km/h; (B) 2,5km/h; (C) 3km/h; (D) 3,5km/h; (E) 4km/h.

Question 5. How many different 4-digit even integers can be form from the elements of the set $\{1, 2, 3, 4, 5\}$.

(A): 4; (B): 5; (C): 8; (D): 9; (E) None of the above.

Question 6. At 3:00 A.M. the temperature was 13^o below zero. By noon it had risen to 32^o. What is the average hourly increase in teparature?

Question 7. Find all integers n such that $60 + 2n - n^2$ is a perfect square.

Question 8. Given a triangle ABC and 2 points $K \in AB$, $N \in BC$ such that $BK = 2AK$, $CN = 2BN$ and Q is the common point of AN and CK. Compute $\frac{S_{\Delta ABC}}{S_{\Delta BCQ}}$.

Question 9. Evaluate the integer part of the number

$$H = \sqrt{1 + 2011^2 + \frac{2011^2}{2012^2}} + \frac{2011}{2012}.$$

Question 10. Solve the following equation

$$\frac{1}{(x+29)^2} + \frac{1}{(x+30)^2} = \frac{13}{36}.$$

Question 11. Let be given a sequence $a_1 = 5$, $a_2 = 8$ and $a_{n+1} = a_n + 3a_{n-1}$, $n = 2, 3, \ldots$. Calculate the greatest common divisor of a_{2011} and a_{2012}.

Question 12. Find all positive integers P such that the sum and product of all its divisors are $2P$ and P^2, respectively.

Question 13. Determine the greatest value of the sum $M = 11xy + 3xz + 2012yz$, where x, y, z are non negative integers satisfying the following condition $x + y + z = 1000$.

Question 14. Let be given a triangle ABC with $\angle A = 90^0$ and the bisectrices of angles B and C meet at I. Suppose that IH is perpendicular to BC (H belongs to BC). If $HB = 5$cm, $HC = 8$cm, compute the area of $\triangle ABC$.

Question 15. Determine the greatest value of the sum $M = xy + yz + zx$, where x, y, z are real numbers satisfying the following condition $x^2 + 2y^2 + 5z^2 = 22$.

1.7.2 Senior Section

Question 1. Let $x = \dfrac{\sqrt{6+2\sqrt{5}} + \sqrt{6-2\sqrt{5}}}{\sqrt{20}}\cdot$ The value of $H = (1 + x^5 - x^7)^{2012^{3^{11}}}$ is

(A): 1; (B): 11; (C): 21; (D): 101; (E) None of the above.

Question 2. Compare the numbers:

$$P = 2^\alpha, \quad Q = 3, \quad T = 2^\beta, \text{ where } \alpha = \sqrt{2}, \beta = 1 + \frac{1}{\sqrt{2}}$$

(A): $P < Q < T$; (B): $T < P < Q$; (C): $P < T < Q$; (D): $T < Q < P$; (E): $Q < P < T$.

Question 3. Let be given a trapezoidal $ABCD$ with the based edges $BC = 3$cm, $DA = 6$cm ($AD \parallel BC$). Then the length of the line EF ($E \in AB$, $F \in CD$ and $EF \parallel AD$) through the common point M of AC and BD is

(A): 3,5cm; (B): 4cm; (C): 4,5cm; (D): 5cm; (E) None of the above.

Question 4. What is the largest integer less than or equal to $4x^3 - 3x$, where $x = \dfrac{1}{2}\left(\sqrt[3]{2+\sqrt{3}} + \sqrt[3]{2-\sqrt{3}}\right)$?

(A): 1; (B): 2; (C): 3; (D): 4; (E) None of the above.

Question 5. Let $f(x)$ be a function such that $f(x)+2f\Big(\dfrac{x+2010}{x-1}\Big) = 4020 - x$ for all $x \neq 1$. Then the value of $f(2012)$ is

(A): 2010; (B): 2011; (C): 2012; (D): 2014; (E) None of the above.

Question 6. For every $n = 2, 3, \ldots,$ we put

$$A_n = \Big(1 - \frac{1}{1+2}\Big) \times \Big(1 - \frac{1}{1+2+3}\Big) \times \cdots \times \Big(1 - \frac{1}{1+2+3+\cdots+n}\Big).$$

Determine all positive integer n ($n \geq 2$) such that $\dfrac{1}{A_n}$ is an integer.

Question 7. Prove that the number $a = \overline{\underbrace{1\ldots1}_{2012}\underbrace{5\ldots5}_{2011}6}$ is a perfect square.

Question 8. Determine the greatest number m such that the system

$$\begin{cases} x^2 + y^2 = 1 \\ |x^3 - y^3| + |x - y| = m^3 \end{cases}$$

has a solution.

Question 9. Let P be the common point of 3 internal bisectors of a given ABC. The line passing through P and perpendicular to CP intersects AC and BC at M and N, respectively. If $AP = 3$cm, $BP = 4$cm, compute the value of $\dfrac{AM}{BN}$?

Question 10. Suppose that the equation $x^3 + px^2 + qx + 1 = 0$, with p, q are rational numbers, has 3 real roots x_1, x_2, x_3, where $x_3 = 2 + \sqrt{5}$, compute the values of p and q?

Question 11. Suppose that the equation $x^3+px^2+qx+r=0$ has 3 real roots x_1, x_2, x_3, where p, q, r are integer numbers. Put $S_n = x_1^n + x_2^n + x_3^n$, $n = 1, 2, \dots$ Prove that S_{2012} is an integer.

Question 12. In an isosceles triangle ABC with the base AB given a point $M \in BC$. Let O be the center of its circumscribed circle and S be the center of the inscribed circle in ΔABC and $SM \parallel AC$. Prove that $OM \perp BS$.

Question 13. A cube with sides of length 3cm is painted red and then cut into $3 \times 3 \times 3 = 27$ cubes with sides of length 1cm. If a denotes the number of small cubes (of 1cm×1cm×1cm) that are not painted at all, b the number painted on one sides, c the number painted on two sides, and d the number painted on three sides, determine the value $a - b - c + d$?

Question 14. Solve, in integers, the equation $16x + 1 = (x^2 - y^2)^2$.

Question 15. Determine the smallest value of the sum $M = xy - yz - zx$, where x, y, z are real numbers satisfying the following condition $x^2 + 2y^2 + 5z^2 = 22$.

1.8 Hanoi Open Mathematics Competition 2013

1.8.1 Junior Section

Question 1. Write 2013 as a sum of m prime numbers. The smallest value of m is:

(A): 2; (B): 3; (C): 4; (D): 1; (E): None of the above.

Question 2. How many natural numbers n are there so that $n^2 + 2014$ is a perfect square.

(A): 1; (B): 2; (C): 3; (D): 4; (E) None of the above.

Question 3. The largest integer not exceeding $[(n+1)\alpha] - [n\alpha]$, where n is a natural number, $\alpha = \dfrac{\sqrt{2013}}{\sqrt{2014}}$, is:
(A): 1; (B): 2; (C): 3; (D): 4; (E) None of the above.

Question 4. Let A be an even number but not divisible by 10. The last two digits of A^{20} are:

(A): 46; (B): 56; (C): 66; (D): 76; (E): None of the above.

Question 5. The number of integer solutions x of the equation below

$$(12x - 1)(6x - 1)(4x - 1)(3x - 1) = 330.$$

is: (A): 0; (B): 1; (C): 2; (D): 3; (E): None of the above.

Short Questions

Question 6. Let ABC be a triangle with area 1 (cm^2). Points D, E and F lie on the sides AB, BC and CA, respectively. Prove that

$$\min\{\text{Area of } \Delta ADF, \text{Area of } \Delta BED, \text{Area of } \Delta CEF\} \leq \frac{1}{4} \ (\text{cm}^2).$$

Question 7. Let ABC be a triangle with $\widehat{A} = 90^0, \widehat{B} = 60^0$ and $BC = 1$cm. Draw outside of ΔABC three equilateral triangles ABD, ACE and BCF. Determine the area of ΔDEF.

Question 8. Let $ABCDE$ be a convex pentagon. Given that

$$\text{area of } \Delta ABC = \text{area of } \Delta BCD = \text{area of } \Delta CDE$$

$$= \text{area of } \Delta DEA = \text{area of } \Delta EAB = 2\text{cm}^2,$$

Find the area of the pentagon.

Question 9. Solve the following system in positive numbers

$$\begin{cases} x + y \leq 1 \\ \dfrac{2}{xy} + \dfrac{1}{x^2 + y^2} = 10. \end{cases}$$

Question 10. Consider the set of all rectangles with a given perimeter p. Find the largest value of

$$M = \frac{S}{2S + p + 2},$$

where S is denoted the area of the rectangle.

Question 11. The positive numbers a, b, c, d, e are such that the following identity hold for all real number x.

$$(x + a)(x + b)(x + c) = x^3 + 3dx^2 + 3x + e^3.$$

Find the smallest value of d.

Question 12. If $f(x) = ax^2 + bx + c$ safisfies the condition

$$|f(x)| < 1, \;\; \forall x \in [-1, 1],$$

prove that the equation $f(x) = 2x^2 - 1$ has two real roots.

Question 13. Solve the system of equations

$$\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{6} \\ \dfrac{3}{x} + \dfrac{2}{y} = \dfrac{5}{6} \end{cases}$$

Question 14. Solve the system of equations

$$\begin{cases} x^3 + y &= x^2 + 1 \\ 2y^3 + z &= 2y^2 + 1 \\ 3z^3 + x &= 3z^2 + 1 \end{cases}$$

Question 15. Denote by $\mathbb{Q}$ and $\mathbb{N}^*$ the set of all rational and positive integer numbers, respectively. Suppose that $\dfrac{ax+b}{x} \in \mathbb{Q}$ for every $x \in \mathbb{N}^*$. Prove that there exist integers A, B, C such that

$$\frac{ax+b}{x} = \frac{Ax+B}{Cx} \text{ for all } x \in \mathbb{N}^*.$$

1.8.2 Senior Section

Question 1. How many three-digit perfect squares are there such that if each digit is increased by one, the resulting number is also a perfect square?

(A): 1; (B): 2; (C): 4; (D): 8; (E) None of the above.

Question 2. The smallest value of the function

$$f(x) = |x| + \left|\frac{1 - 2013x}{2013 - x}\right|$$

where $x \in [-1, 1]$ is

(A): $\frac{1}{2012}$; (B): $\frac{1}{2013}$; (C): $\frac{1}{2014}$; (D): $\frac{1}{2015}$; (E): None of the above.

Question 3. What is the largest integer not exceeding $8x^3 + 6x - 1$, where $x = \frac{1}{2}\left(\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}\right)$?

(A): 1; (B): 2; (C): 3; (D): 4; (E) None of the above.

Question 4. Let $x_0 = [\alpha]$, $x_1 = [2\alpha] - [\alpha]$, $x_2 = [3\alpha] - [2\alpha]$, $x_4 = [5\alpha] - [4\alpha]$, $x_5 = [6\alpha] - [5\alpha]$, $\ldots$, where $\alpha = \frac{\sqrt{2013}}{\sqrt{2014}}$. The value of x_9 is

(A): 2; (B): 3; (C): 4; (D): 5; (E): None of the above.

Question 5. The number n is called a composite number if it can be written in the form $n = a \times b$, where a, b are positive integers greater than 1.

Write number 2013 in a sum of m composite numbers. What is the largest value of m?

(A): 500; (B): 501; (C): 502; (D): 503; (E): None of the above.

Question 6. Let be given $a \in \{0, 1, 2, 3, \ldots, 1006\}$. Find all $n \in \{0, 1, 2, 3, \ldots, 2013\}$ such that $C_{2013}^n > C_{2013}^a$, where $C_m^k = \frac{m!}{k!(m-k)!}$.

Question 7. Let ABC be an equilateral triangle and a point M inside the triangle such that $MA^2 = MB^2 + MC^2$. Draw an equilateral triangle ACD where $D \neq B$. Let the point N inside

ΔACD such that AMN is an equilateral triangle. Determine $\widehat{BMC}$.

Question 8. Let $ABCDE$ be a convex pentagon and

$$\text{area of } \Delta ABC = \text{area of } \Delta BCD = \text{area of } \Delta CDE$$

$$= \text{area of } \Delta DEA = \text{area of } \Delta EAB.$$

Given that area of $\Delta ABCDE = 2$. Evaluate the area of area of ΔABC.

Question 9. A given polynomial $P(t) = t^3 + at^2 + bt + c$ has 3 distinct real roots. If the equation $(x^2 + x + 2013)^3 + a(x^2 + x + 2013)^2 + b(x^2 + x + 2013) + c = 0$ has no real roots, prove that $P(2013) > \dfrac{1}{64}$.

Question 10. Consider the set of all rectangles with a given area S. Find the largest value of

$$M = \frac{16S - p}{p^2 + 2p},$$

where p is the perimeter of the rectangle.

Question 11. The positive numbers a, b, c, d, p, q are such that

$$(x+a)(x+b)(x+c)(x+d) = x^4+4px^3+6x^2+4qx+1 \text{ holds for all real numbers } x.$$

Find the smallest value of p or the largest value of q.

Question 12. The function $f(x) = ax^2 + bx + c$ safisfies the following conditions: $f(\sqrt{2}) = 3$, and

$$|f(x)| \leq 1, \text{ for all } x \in [-1, 1].$$

Evaluate the value of $f(\sqrt{2013})$.

Question 13. Solve the system of equations

$$\begin{cases} xy = 1 \\ \dfrac{x}{x^4+y^2} + \dfrac{y}{x^2+y^4} = 1 \end{cases}$$

Question 14. Solve the system of equations:

$$\begin{cases} x^3 + \frac{1}{3}y & = x^2 + x - \frac{4}{3} \\ y^3 + \frac{1}{4}z & = y^2 + y - \frac{5}{4} \\ z^3 + \frac{1}{5}x & = z^2 + z - \frac{6}{5} \end{cases}$$

Question 15. Denote by $\mathbb{Q}$ and $\mathbb{N}^*$ the set of all rational and positive integral numbers, respectively. Suppose that $\dfrac{ax+b}{cx+d} \in \mathbb{Q}$ for every $x \in \mathbb{N}^*$. Prove that there exist integers A, B, C, D such that

$$\frac{ax+b}{cx+d} = \frac{Ax+B}{Cx+D} \text{ for all } x \in \mathbb{N}^*.$$